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NECESSARY CONDITIONS FOR OPTIMALITY IN ONE FRACTIONAL ORDER DISCRETE CONTROL PROBLEM IN THE PRESENCE OF FUNCTIONAL LIMITATIONS OF THE TYPE INEQUALITIES ON THE RIGHT END OF THE TRAJECTORY

S.T. ALIYEVA [1,2], K.B. MANSIMOV [1,2]

[1]Baku State University,
[2]Institute of Control Systems of Azerbaijan National Academy of Sciences
e-mail: saadata@mail.ru, kamilbmansimov@gmail.com

Abstract: We consider the problem of optimal control of an object described by a system of ordinary non-linear difference equations of fractional order in the presence of functional constraints such as inequalities on the state of the system. A discrete analogue of the Pontryagin maximum principle is established. In the case of convexity of the control domain, a linearized optimality condition is proved.

Keywords: admissible control, optimal control, fractional difference equation, Pontryagin's maximum principle.

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1. Introduction.

In works [1,2] a number of optimal control problems with a free right end were studied, which are described by fractional-order difference equations.

Under various assumptions, a number of necessary optimality conditions are established.

In the proposed work, we study the case of the presence of functional constraints of the type of inequalities.

Necessary optimality conditions of the type [5] are established. As noted in [5], such necessary optimality conditions are, in contrast to the classical optimality conditions, of a constructive nature.

The results obtained allow us to study the cases of their degeneracy.

Formulation of the problem.

Consider the problem of minimizing the functional

$$S_0(u) = \varphi_0\big(x(t_1)\big), \tag{1}$$

under restrictions

$$S_i(u) = \varphi_i\big(x(t_1)\big) \le 0, i = \overline{1,p}, \tag{2}$$

$$\Delta^\alpha x(t+1) = f\big(t, x(t), u(t)\big), t \in T = \{t_0, t_0+1, \dots, t_1-1\}, \tag{3}$$

$$x(t_0) = x_0, \tag{4}$$

$$u(t) \in U \subset R^r, \quad t \in T. \tag{5}$$

Here $\varphi_i(x), i = \overline{0,p}$ are given continuously differentiable scalar functions, $u(t)$ $r-$ is an r-dimensional discrete vector of control actions, U is a given nonempty and bounded set, the numbers t_o, t_1 and the constant vector $x_0 -$ are given, $f(t,x,u) -$ is a given n-dimensional vector function continuous in the set of variables together with partial derivatives with respect to x, a $\Delta^\alpha x(t), 0 < \alpha \leq 1$- fractional operator of order α (see e.g. [3-13]).

A control $u(t) = \{u(t_0), u(t_0+1), \ldots, u(t_1-1)\}$ is called an admissible control in problem (1)-(5) if the corresponding solution $x(t)$ of the system (3)-(4) satisfies the constraints (2).

2. Quality criterion increment formula.

Let $\big(u(t), x(t)\big)$ be fixed and a $\big(\bar{u}(t) = u(t) + \Delta u(t), \bar{x}(t) = x(t) + \Delta x(t)\big)$ – arbitrary - admissible processes.

Suppose that, in problem (1.1)-(1.5) and along the admissible process $x(t), u(t)$ it is the set of admissible velocities of the system (1.3)-(1.4) i.e. lots of

$$f(t, x(t), U) = \{y \in R^n : y = f(t, x(t), v), v \in U, t \in T\} \tag{2.1}$$

convex.

Let's put

$$I(u) = \big\{i : \varphi_i\big(x(t_1)\big) = 0, i = \overline{1,p}\big\}, J(u) = \{0\} \cup I(u).$$

In what follows, for the prostate, we will assume that

$$I(u) = \{1,2,\ldots,m\} \ (m < p).$$

From the introduced notation, it is clear that $\Delta x(t)$ ((trajectory increment $x(t)$), corresponding to $\Delta u(t)$ (control increment $u(t)$) will satisfy the system

$$\Delta^\alpha\big(\Delta x(t+1)\big) = f\big(t, \bar{x}(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big), \tag{2.2}$$

$$\Delta x(t_0) = 0. \tag{2.3}$$

Let us calculate the increment of the functional $S_i(u)$ corresponding to the admissible controls $u(t)$ and $\bar{u}(t) = u(t) + \Delta u(t)$:

$$\begin{aligned} \Delta S_i(u) &= S_i(\bar{u}) - S_i(u) = S_i\big(u + \Delta u(t)\big) - S_i(u) = \\ &= \varphi_i\big(x(t_1) + \Delta x(t_1)\big) - \varphi_i\big(x(t_1)\big). \end{aligned} \tag{2.4}$$

Denote by $\psi_i(t)$ the so far unknown n-dimensional column vectors and introduce analogues of the Hamilton-Pontryagin function in the form

$$H(t, x, u, \psi_i) = \psi_i{}'(t) f(t, x, u).$$

Multiplying both parts of relation (2.2) scalarly by $\psi_i(t)$, and then summing both parts of the resulting identity over t from t_0 to $t_1 - 1$ and taking into account the expression of the Hamilton-Pontryagin function, we get that

$$\begin{aligned} \sum_{t=t_0}^{t_1-1} \psi_i{}'(t)\, \Delta^\alpha\big(\Delta x(t+1)\big) &= \sum_{t=t_0}^{t_1-1} \psi_i{}'(t) \left[f\big(t, \bar{x}(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big)\right] = \\ &= \sum_{t=t_0}^{t_1-1} \left[H\big(t, \bar{x}(t), \bar{u}(t), \psi_i(t)\big) - H\big(t, x(t), u(t), \psi_i(t)\big)\right]. \end{aligned} \tag{2.5}$$

Taking into account this identity, the increment (2.4) of the functional $S_i(u)$ can be represented as

$$\Delta S_i(u) = \varphi_i\big(x(t_1) + \Delta x(t_1)\big) - \varphi_i\big(x(t_1)\big) + \sum_{t=t_0}^{t_1-1} \psi_i{}'(t)\, \Delta^\alpha\big(\Delta x(t+1)\big) -$$

$$- \sum_{t=t_0}^{t_1-1} \big[H\big(t, \bar{x}(t), \bar{u}(t), \psi_i(t)\big) - H\big(t, x(t), u(t), \psi_i(t)\big)\big]. \qquad (2.6)$$

Now let's deal with the transformation of the left side of the term in formula (2.5). To this end, consider the expression

$$\sum_{t=t_0}^{t_1-1} \psi_i{}'(t)\, \Delta^\alpha\big(\Delta x(t+1)\big)$$

Having made the change of variables $t + 1 = s$ in it and taking into account the initial condition (2.3), we obtain

$$\sum_{t=t_0}^{t_1-1} \psi_i{}'(t)\, \Delta^\alpha\big(\Delta x(t+1)\big) = \sum_{t=t_0+1}^{t_1} \psi_i{}'(t-1)\, \Delta^\alpha\big(\Delta x(t)\big) =$$

$$= \psi_i{}'(t_1 - 1)\Delta^\alpha\big(\Delta x(t_1)\big) - \psi_i{}'(t_0 - 1)\Delta^\alpha\big(\Delta x(t_0)\big) + \sum_{t=t_0}^{t_1-1} \psi_i{}'(t-1)\, \Delta^\alpha\big(\Delta x(t)\big)$$

$$=$$

$$= \psi_i{}'(t_1 - 1)\Delta^\alpha\big(\Delta x(t_1)\big) + \sum_{t=t_0}^{t_1-1} \psi_i{}'(t-1)\, \Delta^\alpha\big(\Delta x(t)\big). \qquad (2.7)$$

Further, taking into account the theorem on fractional summation (see, for example, [9]) given above in parts, we have

$$\sum_{t=t_0}^{t_1-1} \psi_i{}'(t-1)\, \Delta^\alpha\big(\Delta x(t)\big) = \psi_i{}'(t_1 - 1)\Delta x(t_1) - \psi_i{}'(t_0 - 1)\Delta x(t_0) +$$

$$+ \sum_{t=t_0}^{t_1-2} \Delta^\alpha \psi_i(t-1)\Delta x(t) + \frac{\mu}{\Gamma(\mu)} \Delta x(t_0) \times$$

$$\times \left(\sum_{t=t_0}^{t_1-1} (t + \mu - t_0)^{(\mu-1)} \psi_i\,(t) - \sum_{t=\sigma(a)}^{t_1-1} \big(t + \mu - \sigma(t_0)\big)^{(\mu-1)}\, \Delta x(t) \right) =$$

$$= \psi_i{}'(t_1 - 1)\Delta x(t_1) + \sum_{t=t_0}^{t_1-2} \Delta^\alpha \psi_i(t-1)\Delta x(t). \qquad (2.8)$$

Taking into account identity (2.8), from the increment formula (2.6) we obtain

$$\Delta S_i(u) = S_i(\bar{u}) - S_i(u) = \varphi_i\big(x(t_1) + \Delta x(t_1)\big) - \varphi_i\big(x(t_1)\big) +$$
$$+\psi_i{}'(t_1 - 1)\Delta x(t_1) + \sum_{t=t_0}^{t_1-2} \Delta^\alpha \psi_i(t-1)\Delta x(t) -$$
$$-\sum_{t=t_0}^{t_1-1} \big[H\big(t, \bar{x}(t), \bar{u}(t), \psi_i(t)\big) - H\big(t, x(t), u(t), \psi_i(t)\big)\big]. \qquad (2.9)$$

In what follows, we will use the notation

$$H^{(i)}{}_x[t] \equiv \psi_i{}'(t) f_x\big(t, x(t), u(t)\big),$$
$$\Delta_{\bar{u}(t)} f[t] \equiv f\big(t, x(t), \bar{u}(t)\big) - f\big(t, x(t), u(t)\big),$$
$$\Delta_v H^{(i)}[t] \equiv \psi_i{}'(t) \Delta_v f\big(t, x(t), u(t)\big).$$

Under the assumptions made, the increment formula (2.9) of the functional $S_i(u)$, using the Taylor formula, corresponding to the admissible controls $\bar{u}(t)$ and $u(t)$ can be represented as:

$$\Delta S_i(u) = \frac{\varphi'_i\big(x(t_1)\big)}{\partial x}\Delta x(t_1) + \psi_i{}'(t_1 - 1)\Delta x(t_1) +$$
$$+\sum_{t=t_0}^{t_1-1} \psi_i{}'(t-1)\Delta x(t) - \sum_{t=t_0}^{t_1-2} \Delta^\alpha \psi_i{}'(t-1)\Delta x(t) -$$
$$-\sum_{t=t_0}^{t_1-1} H^{(i)}{}'_x[t]\Delta x(t) - \sum_{t=t_0}^{t_1-1} \Delta_v H^{(i)}[t] - \sum_{t=t_0}^{t_1-1} \Delta_v H^{(i)}{}_x[t]\Delta x(t)$$
$$+o_1{}^i(\|\Delta x(t_1)\|) - \sum_{t=t_0}^{t_1-1} o_2{}^i(\|\Delta x(t)\|). \qquad (2.10)$$

Here $\|\alpha\|$ is the norm of the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)'$ defined by the formula

$$\|\alpha\| = \sum_{i=1}^{n} |\alpha_i|,$$

a $o(\alpha)$ is a value of a higher order than α, i.e. $o(\alpha)/_\alpha \to 0$, as $\alpha \to 0$.

Now suppose that $\psi_i(t)$ is the solution of the following system of linear homogeneous fractional order difference equations

$$\begin{cases} \sum_{t=t_0}^{t_1-2} \Delta^{\alpha}\psi_i{}'(t-1) = H^{(i)}{}'_x[t], \qquad t = t_1-1, t_1-2, \dots, t_0, \\ \psi_i(t_1-1) = -\dfrac{\varphi_i(x(t_1))}{\partial x}. \end{cases} \tag{2.11}$$

System (2.11) is called the adjoint system in problem (1.1)–(1.5) under consideration.

When relations (2.11) are fulfilled, the increment formula (2.10) will take the following form

$$\Delta S_i(u) = -\sum_{t=t_0}^{t_1-1} \Delta_v H^{(i)}[t] + \eta_i(u, \Delta u). \tag{2.12}$$

where

$$\eta_i(u,\Delta u) = o_1{}^i \|\Delta x(t_1)\| -$$
$$-\sum_{t=t_0}^{t_1-1} o_2{}^i \|\Delta x(t)\| - \sum_{t=t_0}^{t_1-1} \Delta_v H^{(i)}{}_x[t]\Delta x(t). \tag{2.13}$$

Let $\varepsilon \in [0,1]$ be an arbitrary number, and $v(t) \in U, t \in T$ an arbitrary vector of control actions.

Due to the convexity of the set (6), the special increment of the admissible control $u(t)$ can be defined as follows:

$$\Delta u_\varepsilon(t) = v(t,\varepsilon) - u(t) \tag{2.14}$$

Here $\varepsilon \in [0,1] -$ is an arbitrary number, $v(t,\varepsilon) \in U, t \in T$ is an admissible control such that

$$\Delta_{v(t,\varepsilon)} f[t] = \varepsilon \Delta_{v(t)} f[t].$$

Denote by $\Delta x(t;\varepsilon)$ the special increment of the admissible trajectory $x(t)$, corresponding to the special increment (2.14) of the control $u(t)$.

Taking into account the estimate from [9] , we obtain that

$$\|\Delta x(t;\varepsilon)\| \le L_3\varepsilon, \quad t \in T \cup t_1, \tag{2.15}$$
$$(L_3 = const > 0.$$

Taking into account the estimate from (2.15) for $\eta_i(u,\Delta u)$ we obtain that

$$\eta_i(u,\Delta u_\varepsilon) = o(\varepsilon).$$

Therefore, the increment formula (2.12) implies the expansion

$$S_i\big(u(t) + \Delta u_\varepsilon(t)\big) - S_i\big(u(t)\big) = -\varepsilon \sum_{t=t_0}^{t_1-1} \Delta_v H^{(i)}[t] + o(\varepsilon). \tag{2.16}$$

Has it

Theorem 2.1. If along an admissible process $\big(u(t), x(t)\big)$ the set of admissible velocities of system (3) is convex, then for the admissible control $u(t)$ to be optimal in problem (1)-(5) it is necessary that the inequality

$$\min_{i\in J(u)} \sum_{t=t_0}^{t_1-1} \Delta_v H^{(i)}[t] \le 0. \tag{2.17}$$

performed for all $\upsilon(t)\in U$, $t\in T$.

Proof: Assume the opposite. Let an admissible control $u(t)$ be optimal and condition (2.17) not be satisfied, i.e. there exists $\bar{v}(t) \in U$ such that

$$\min_{i\in J(u)} \sum_{t=t_0}^{t_1-1} \Delta_{\bar{v}} H^{(i)}[t] > 0. \tag{2.18}$$

The special increment of the optimal control $u(t)$ is determined by the formula

$$\Delta\bar{u}_\varepsilon(t) = \bar{v}(t,\varepsilon) - u(t),$$

where $\varepsilon \in [0,1]$, a $\bar{v}(t,\varepsilon) \in U,\ t \in T,$ a vector such that

$$\Delta_{\bar{v}(t,\varepsilon)} f[t] = \varepsilon \Delta_{\bar{v}(t)} f[t], \bar{v}(t) \in U, t \in T.$$

Consequently, inequality (2.16) can be written in the following form

$$S_i\big(u(t) + \Delta\bar{u}_\varepsilon(t)\big) - S_i\big(u(t)\big) =$$

$$= -\varepsilon \sum_{t=t_0}^{t_1-1} \Delta_{\bar{v}(t)} H^{(i)}[t] + o(\varepsilon), \qquad i = \overline{0,p} \tag{2.19}$$

Hence, taking into account the structure of the set $I(u)$ and (2.18) we obtain that for sufficiently small ε, for all $i \in I(u)$

$$S_i\big(u(t) + \Delta u_\varepsilon(t)\big) - S_i\big(u(t)\big) = -\varepsilon \sum_{t=t_0}^{t_1-1} \Delta_{v(t)} H^{(i)}[t]\Delta x(t) + o(\varepsilon) < 0.$$

Further, for $i \in \{\overline{1,p}\}\backslash I(u)$

$$S_i\big(u(t) + \Delta u_\varepsilon(t)\big) - S_i\big(u(t)\big) = \varphi_i(x(t_1) + \Delta x_\varepsilon(t_1)) < 0,$$

and besides

$$S_0\big(u(t) + \Delta u_\varepsilon(t)\big) < S_0(u(t)).$$

The latter contradicts the optimality of the control $u(t)$.

This proves the theorem.

Let in problem (1.1)-(1.5) the set U be convex, and $f(t,x,u)$ be continuous in the set of variables together with $f_x(t,x,u)$ and $f_u(t,x,u)$.

By analogy with the proof of Theorem (2.1), we prove

Theorem 2.2 If in problem (1.1)-(1.5) the set U is convex, and $f(t,x,u)$ is continuous in the set of variables together with partial derivatives in (x,u), then for the optimality of the admissible control $u(t)$ it is necessary that inequality

$$\min_{i\in J(u)} \sum_{t=t_0}^{t_1-1} H_u{}^{(i)\prime}[t](v(t)-u(t)) \leq 0. \tag{2.20}$$

held for all $v(t) \in U, t \in T$.

The optimality condition (13) is an analogue of the linearized maximum principle for the problem under consideration.

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